Linear Preservers of Majorization on $\ell^p(I)$

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Abstract

In this paper, using doubly stochastic operators, we extend the notion of majorization to the space $\ell^p(I)$, where I is assumed to be an infinite set, and then, in the case $p \in (1, \infty)$, characterize the structure of all bounded linear maps on this space which preserve majorization.

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1 Introduction

Majorization in finite dimension has been widely studied as a result of its applications to many areas of mathematics, such as matrix analysis, operator theory, frame theory, and inequalities involving convex functions, as well as other sciences like physics and economics. See, for example, the papers [2], [3], [7] and [8]. We also refer the reader to the standard text by Marshall and Olkin [6]. For a pair of vectors x and y in $\mathbb{R}^n$, x is called majorized by y, denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \le \sum_{i=1}^{k} y_i^{\downarrow} \qquad (k = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$$

where $x_1^{\downarrow} \ge x_2^{\downarrow} \ge \cdots \ge x_n^{\downarrow}$ is the decreasing rearrangement of components of a vector x.

There are some equivalent conditions for vector majorization. For example, Hardy, Littlewood and Polya [4] proved that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic

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matrix D. We recall that a square matrix with non-negative real entries is called doubly stochastic if each of its row sums and column sums equal 1. As we will see in Section 3, this equivalent condition will serve as our motivation to define majorization on certain spaces other than $\mathbb{R}^n$.

In more recent years the extension of majorization theory to infinite sequences has turned up and obtained some applications (see for example [5]). In this paper, we will consider majorization on the space $\ell^p(I)$, for $1 \leq p < +\infty$, and in the case where I is an infinite set. Our main interest is in linear maps which preserve majorization. The following result, due to Ando, characterizes these maps in finite dimension.

Theorem 1.1 [1]. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then $T(x) \prec T(y)$ whenever $x \prec y$ (i.e. T preserves majorization) if and only if one of the following conditions hold.*
(i) $T(x) = \mathrm{tr}(x)a$, *for some* $a \in \mathbb{R}^n$.
(ii) $T(x) = \beta P(x) + \gamma \mathrm{tr}(x)e$ *for some* $\beta, \gamma \in \mathbb{R}$ *and permutation* $P : \mathbb{R}^n \to \mathbb{R}^n$.

Here $\mathrm{tr}(x) = \sum_{i=1}^n x_i$ is the trace of the vector $x \in \mathbb{R}^n$. Also $e \in \mathbb{R}^n$ denotes the vector $(1, 1, \ldots, 1)$.

Quit different from this result, our main theorem asserts that if I is an infinite set and $1 < p < +\infty$, then a linear map $T : \ell^p(I) \to \ell^p(I)$ preserves majorization if and only if the columns of T are permutations of each other and in each row of T there is at most one non-zero element. Note that, in condition (ii) of Theorem 1.1, if $\gamma = 0$ then the resulted T has the structure mentioned above.

The organization of the paper is as follows. In the next section we recall the definition of doubly stochastic operators on the space $\ell^p(I)$, for $1 \leq p < +\infty$. We obtain some properties and give a way of constructing these operators. In section 3, we give a definition of majorization on $\ell^p(I)$ based on doubly stochastic operators. The main theorem of this section asserts that if $f \prec g$ and $g \prec f$, for $f, g \in \ell^p(I)$, then there exists a permutation $P : \ell^p(I) \to \ell^p(I)$ such that $f = Pg$, a result which is well-known if I is a finite set. Finally, in the last section we characterize the linear preservers of majorization on $\ell^p(I)$, for an infinite set I and in the case where $1 < p < +\infty$. We end this section with an example which shows that this characterization is not true for $p = 1$.

2 Doubly Stochastic Operators

We first recall some definitions. For a non-empty set I and a real $p \in [1, +\infty)$, let $\ell^p(I)$ be the Banach space of all functions $f : I \to \mathbb{R}$ with

$$\|f\|_p := \big(\sum_{i \in I} |f(i)|^p\big)^{\frac{1}{p}} < +\infty$$

An element $f \in \ell^p(I)$ can be represented as $\sum_{i \in I} f(i)\, e_i$, where $e_i : I \to \mathbb{R}$ is defined by $e_i(j) = \delta_{ij}$, the Kroneker delta. Considering e_i as an element of the dual space of $\ell^p(I)$, we have

$$\forall i \in I \qquad f(i) = \langle f, e_i \rangle$$

where $\langle \cdot, \cdot \rangle$ stands for the dual pairing. Hence, for $f \in \ell^p(I)$ we will have the representation

$$f = \sum_{i \in I} \langle f, e_i \rangle e_i$$

It is a well-known fact that $\ell^p(I)$ is an ordered vector space (and, in fact, a Banach lattice) under the natural partial ordering on the set of real valued functions defined on I. We recall that a linear operator A on an ordered vector space X is called positive if $Ax \geq 0$ whenever $x \geq 0$.

Definition 2.1 *Let I and J be two non-empty sets, and suppose $A : \ell^p(J) \to \ell^p(I)$ is a bounded linear operator. Then A is called*

(i) *row stochastic (respectively, column stochastic) if A is positive and*

$$\forall i \in I, \quad \sum_{j\in J} \langle Ae_j, e_i\rangle = 1 \quad \Big(\forall j \in J, \quad \sum_{i\in I} \langle Ae_j, e_i\rangle = 1\Big) \tag{1}$$

(ii) *doubly stochastic if A is both row and column stochastic.*

(iii) *a permutation if there exists a bijection $\theta : J \to I$ for which $Ae_j = e_{\theta(j)}$, for each $j \in J$.*

As the following theorem shows if there exists a doubly stochastic operator between the spaces $\ell^p(I)$ and $\ell^p(J)$ then I and J have the same cardinality. This result plays a crucial role in the proof of the main theorem of Section 3.

Theorem 2.2 *Let I, J be two arbitrary non-empty sets. Then there exists a doubly stochastic operator $D : \ell^p(J) \to \ell^p(I)$ if and only if $|J| = |I|$, where $|I|$ denotes the cardinal number of a set I.*

Proof. First, suppose there exists a doubly stochastic operator $D : \ell^p(J) \to \ell^p(I)$. Using the relation

$$\sum_{j\in J} 1 = \sum_{j\in J}\sum_{i\in I} \langle De_j, e_i\rangle = \sum_{i\in I}\sum_{j\in J} \langle De_j, e_i\rangle = \sum_{i\in I} 1,$$

J is finite if and only if I is finite, and in this case $|I| = |J|$.

Now suppose J is infinite. Let

$$C = \{(i, j) \in I \times J \ ; \ \langle De_j, e_i\rangle > 0\}.$$

Then $C = \bigcup_{i\in I}(\{i\} \times C_i) = \bigcup_{j\in I}(C^j \times \{j\})$, where $C_i = \{j \in J \ ; \ \langle De_j, e_i\rangle > 0\}$ and $C^j = \{i \in I \ ; \ \langle De_j, e_i\rangle > 0\}$. Note that since D is doubly stochastic, each C_i and C^j is non-empty and at most countable. Moreover, $C_i \cap C_{i'} = \emptyset$ and $C^j \cap C^{j'} = \emptyset$ for distinct $i, i' \in I$ and distinct $j, j' \in J$. Hence

$$|I| \leq |C| \leq \aleph_0 \times |I| \quad , \quad |J| \leq |C| \leq \aleph_0 \times |J|$$

where $\aleph_0$ is the cardinal number of $\mathbb{N}$. Since $|I|, |J| \geq \aleph_0$, we have also $\aleph_0 \times |I| = |I|$ and $\aleph_0 \times |J| = |J|$. Therefore $|I| = |C| = |J|$.

Conversely, let $\theta : J \to I$ be a bijection. If $D : \ell^p(J) \to \ell^p(I)$ is defined for each $f = \sum_{j\in J} f(j)e_j \in \ell^p(J)$ by $Df = \sum_{j\in J} f(j)e_{\theta(j)}$, then it is easily verified that D is doubly

stochastic. $\square$

Since in this paper we are going to work with doubly stochastic operators, according to the previous theorem, we may assume that $I = J$. The set of all row stochastic, column stochastic, doubly stochastic operators and permutation maps on $\ell^p(I)$ are denoted, respectively, by $\mathcal{RS}\big(\ell^p(I)\big)$, $\mathcal{CS}\big(\ell^p(I)\big)$, $\mathcal{DS}\big(\ell^p(I)\big)$ and $\mathcal{P}\big(\ell^p(I)\big)$. It is easily seen that $\mathcal{P}\big(\ell^p(I)\big) \subset \mathcal{DS}\big(\ell^p(I)\big)$. To obtain an essential property of these sets of operators, we need the following lemma.

Lemma 2.3 *Let $p \in [1, +\infty)$ and $A : \ell^p(I) \to \ell^p(I)$ be a positive bounded linear operator. Then*

(i) *A is row stochastic if and only if*

$$\forall f \in \ell^1(I), \qquad \sum_{j\in I} \langle Ae_j, f\rangle = \sum_{i\in I} f(i) \tag{2}$$

(ii) *A is column stochastic if and only if*

$$\forall f \in \ell^1(I), \qquad \sum_{i\in I} \langle Af, e_i\rangle = \sum_{i\in I} f(i) \tag{3}$$

Proof. (i) Let $A : \ell^p(I) \to \ell^p(I)$ be row stochastic. Suppose $q \in (1, +\infty]$ is the exponent conjugate of p. Using the inclusion $\ell^1(I) \subset \ell^q(I)$, if $f \in \ell^1(I)$ then the map $\langle \cdot, f\rangle : \ell^p(I) \to \mathbb{R}$ is a bounded linear functional. Moreover, if $f = \sum_{i\in I} f(i)e_i$ then $\langle \cdot, f\rangle = \sum_{i\in I} f(i)\langle \cdot, e_i\rangle$. To prove this last equality, it suffices to consider $\ell^p(I)$ as a subset of $\ell^\infty(I) = \big(\ell^1(I)\big)^*$.

Since

$$\sum_{i\in I}\sum_{j\in I} |f(i)|\langle Ae_j, e_i\rangle = \sum_{i\in I} |f(i)| < +\infty,$$

by Fubini's Theorem, we have

$$\sum_{j\in I} \langle Ae_j, f\rangle = \sum_{j\in I}\sum_{i\in I} f(i)\langle Ae_j, e_i\rangle = \sum_{i\in I}\sum_{j\in I} f(i)\langle Ae_j, e_i\rangle = \sum_{i\in I} f(i)$$

The converse is clear.
(ii) Suppose A is column stochastic. Let $A^* : \ell^q(I) \to \ell^q(I)$ be the adjoint map. It is easily seen that A^* is row stochastic. Hence, by part (i),

$$\forall f \in \ell^1(I), \qquad \sum_{i\in I} \langle e_i, Af\rangle = \sum_{i\in I} \langle A^* e_i, f\rangle = \sum_{i\in I} f(i)$$

$\square$

Theorem 2.4 *If A and B belong to $\mathcal{RS}\big(\ell^p(I)\big)$ then so does AB, i.e. the set $\mathcal{RS}\big(\ell^p(I)\big)$ is closed under combination. The same conclusion holds for sets $\mathcal{CS}\big(\ell^p(I)\big)$ and $\mathcal{DS}\big(\ell^p(I)\big)$.*

Proof. Let $A, B \in \mathcal{RS}\big(\ell^p(I)\big)$ and suppose A^* is the adjoint of A. Then, using Lemma 2.3, for $i \in I$ we have

$$\begin{aligned}\sum_{j\in I}\langle ABe_j, e_i\rangle &= \sum_{j\in I}\langle Be_j, A^*e_i\rangle\\ &= \sum_{r\in I}\langle A^*e_i, e_r\rangle\\ &= \sum_{r\in I}\langle e_i, Ae_r\rangle = 1\end{aligned}$$

i.e. $AB \in \mathcal{RS}\big(\ell^p(I)\big)$. □

Lemma 2.5 *If $D \in \mathcal{DS}\big(\ell^p(I)\big)$, then $\|D\| \le 1$.*

Proof. For $f = \sum_{j\in I} f(j)e_j \in \ell^p(I)$, using the continuity of D, we have

$$Df = \sum_{j\in I} f(j)De_j$$

Hence

$$\begin{aligned}\|Df\|_p^p &= \sum_{i\in I}\big|\langle Df, e_i\rangle\big|^p\\ &= \sum_{i\in I}\big|\sum_{j\in I} f(j)\langle De_j, e_i\rangle\big|^p\\ &\le \sum_{i\in I}\sum_{j\in I}|f(j)|^p\langle De_j, e_i\rangle\end{aligned}$$

The last inequality has been resulted from Jensen's inequality and the fact that D is row stochastic. Now changing the order of summation, and using the fact that D is also column stochastic, we have

$$\|Df\|_p^p \le \sum_{j\in I}|f(j)|^p\sum_{i\in I}\langle De_j, e_i\rangle = \|f\|_p^p$$

from which the result follows. □

The following proposition, which presents a simple way to construct doubly stochastic operators, will be used in next sections.

Proposition 2.6 *Let I be a non-empty set and $p \in [1,\infty)$. Then corresponding to a family of non-negative real numbers $\{d_{ij}\ ;\ i, j \in I\}$ with*

$$\forall i \in I, \quad \sum_{j\in I} d_{ij} = 1, \qquad \forall j \in I, \quad \sum_{i\in I} d_{ij} = 1 \tag{4}$$

there exists a unique doubly stochastic operator D on $\ell^p(I)$ such that

$$\langle De_j, e_i\rangle = d_{ij}.$$

Proof. Let $\{d_{ij}; i,j \in I\}$ be a family of non-negative and real numbers which satisfy (4) and suppose $f = \sum_{j\in I} f(j)e_j$ is any arbitrary element of $\ell^p(I)$. For $1 \le p < \infty$, from Jensen's inequality, we have

$$|\sum_{j\in I} f(j)d_{ij}|^p \le \sum_{j\in I} |f(j)|^p d_{ij}$$

which holds for each $i \in I$. Thus

$$\sum_{i\in I} |\sum_{j\in I} f(j)d_{ij}|^p \le \sum_{i\in I}\sum_{j\in I} |f(j)|^p d_{ij} = \sum_{j\in I}\sum_{i\in I} |f(j)|^p d_{ij} = \|f\|^p.$$

Hence the linear operator $D : \ell^p(I) \to \ell^p(I)$ defined by

$$Df = \sum_{i\in I} \Big(\sum_{j\in I} f(j)d_{ij}\Big) e_i$$

is bounded.
Since $\langle De_j, e_i\rangle = d_{ij}$ for each $i,j \in I$, by assumption we have $D \in \mathcal{DS}\big(\ell^p(I)\big)$.
To show the uniqueness of D, suppose $A : \ell^p(I) \to \ell^p(I)$ is a bounded linear operator which satisfies $\langle Ae_j, e_i\rangle = d_{ij}$, for all $i,j \in I$. For each $i \in I$, and $f = \sum_{j\in I} f(j)e_j \in \ell^p(I)$, we have

$$(Af)(i) = \langle Af, e_i\rangle = \sum_{j\in I} f(j)\langle Ae_j, e_i\rangle = \sum_{j\in I} d_{ij} f(j) = \langle Df, e_i\rangle = (Df)(i).$$

Thus $A = D$. □

3 Majorization on $\ell^p(I)$

As was pointed out in the Introduction, the notion of majorization in finite dimension has several equivalents, each of which can be used to extend this theory to more general spaces. Here, we take the approach based on the doubly stochastic operators.

Definition 3.1 *For two elements $f, g \in \ell^p(I)$, we say f is majorized by g (or g majorizes f), and denote it by $f \prec g$, if there exists a doubly stochastic operator $D \in \mathcal{DS}\Big(\ell^p(I)\Big)$ such that $f = Dg$.*

In order to obtain some consequences of this definition we need the following lemma.

Lemma 3.2 *Let $-\infty \le a < b \le +\infty$ and $\phi : (a,b) \to [0, +\infty)$ be a convex function. For $f, g \in \ell^p(I)$ with $Im(f), Im(g) \subseteq (a,b)$, if $f \prec g$ then*

$$\sum_{i\in I} \phi(f_i) \le \sum_{i\in I} \phi(g_i), \tag{5}$$

where $f_i = f(i)$ and $g_i = g(i)$ for all $i \in I$.

Proof. Suppose $f = Dg$, for some $D \in \mathcal{DS}\Big(\ell^p(I)\Big)$. Hence, for each $i \in I$,

$$f_i = \langle f, e_i \rangle = \sum_{j \in I} \langle De_j, e_i \rangle g_j$$

Since ϕ is continuous and convex we will obtain

$$\phi(f_i) \leq \sum_{j \in I} \langle De_j, e_i \rangle \phi(g_j)$$

Thus

$$\begin{aligned} \sum_{i \in I} \phi(f_i) &\leq \sum_{i \in I} \sum_{j \in I} \langle De_j, e_i \rangle \phi(g_j) \\ &= \sum_{j \in I} \sum_{i \in I} \langle De_j, e_i \rangle \phi(g_j) \\ &= \sum_{j \in I} \phi(g_j) \end{aligned}$$

$\square$

Corollary 3.3 *For $f, g \in \ell^p(I)$, if $f \prec g$ and $g \prec f$ then*

$$\sum_{i \in I} \phi(f_i) = \sum_{i \in I} \phi(g_i), \tag{6}$$

for every convex function $\phi : (a, b) \to [0, +\infty)$ with $Im(f), Im(g) \subseteq (a, b)$.

It must be noted that the converse of this corollary is not true in general.

Example 3.4 For $f = \sum_{n \in \mathbb{N}} \frac{1}{2^n} e_{n+1}$ and $g = \sum_{n \in \mathbb{N}} \frac{1}{2^n} e_n$ in $\ell^p(\mathbb{N})$, let $\phi : (a, b) \to [0, \infty)$ be a convex function with $Im(f), Im(g) \subseteq (a, b)$. First, suppose $\phi(0) > 0$. Then

$$\lim_{n \to \infty} \phi(f_n) = \lim_{n \to \infty} \phi(g_n) = \phi(0) > 0,$$

which shows that

$$\sum_{n \in \mathbb{N}} \phi(f_n) = \sum_{n \in \mathbb{N}} \phi(g_n) = +\infty.$$

If $\phi(0) = 0$ then

$$\sum_{n \in \mathbb{N}} \phi(f_n) = \sum_{n \in \mathbb{N}} \phi(\frac{1}{2^n}) = \sum_{n \in \mathbb{N}} \phi(g_n) < +\infty.$$

Hence for every convex function $\phi : (a, b) \to [0, +\infty)$ we have $\sum_{n \in \mathbb{N}} \phi(f_n) = \sum_{n \in \mathbb{N}} \phi(g_n)$.

Now if for some doubly stochastic $D \in \mathcal{DS}\big(\ell^p(I)\big)$, $f = Dg$, then the equality

$$0 = f_1 = \sum_{n \in \mathbb{N}} \langle De_n, e_1 \rangle g_n = \sum_{n \in \mathbb{N}} \frac{\langle De_n, e_1 \rangle}{2^n}$$

implies $\langle De_n, e_1 \rangle = 0$, for all $n \in \mathbb{N}$. Thus $\sum_{n \in \mathbb{N}} \langle De_n, e_1 \rangle = 0$ which contradicts the fact that D is doubly stochastic. Hence $f \not\prec g$. A similar argument shows even $g \not\prec f$.

The following theorem, which is our main result in this section, will play a crucial rule in the next section.

Theorem 3.5 *For $f, g \in \ell^p(I)$ the following conditions are equivalent.*

(1) $f \prec g$ *and* $g \prec f$.

(2) *there is a permutation* $P \in \mathcal{P}\Big(\ell^p(I)\Big)$ *such that* $f = Pg$.

Proof. For each $f \in \ell^p(I)$ let I_f^+, I_f^0 and I_f^- be defined as follows.

$$\begin{aligned} I_f^+ &= \{i \in I; f(i) > 0\}, \\ I_f^0 &= \{i \in I; f(i) = 0\}, \\ I_f^- &= \{i \in I; f(i) < 0\}. \end{aligned}$$

It is clear that both I_f^+ and I_f^- are at most countable. Let $\{I_f^n; n \in \mathbb{N}\}$ be a family of subsets of I_f^+ defined inductively as follows.

$$I_f^1 := \Big\{i \in I_f^+; f(i) = \max\{f(j); j \in I_f^+\}\Big\}$$

and for $n > 1$,

$$I_f^n := \Big\{i \in I_f^+; f(i) = \max\{f(j); j \in I_f^+ \smallsetminus \bigcup_{k=1}^{n-1} I_f^k\}\Big\}$$

It is easily seen that I_f^n is at most a finite set, and that if I_f^+ is infinite, then $I_f^n \neq \emptyset$, for each $n \in \mathbb{N}$. Moreover, the family $\{I_f^n; n \in \mathbb{N}\}$ is mutually disjoint and $I_f^+ = \cup_{n \in \mathbb{N}} I_f^n$. For $n \in \mathbb{N}$ with $I_f^n \neq \emptyset$, let $f_n > 0$ be the value of f on I_f^n. If $I_f^n = \emptyset$ then we define f_n equal 0. It is clear that for $m, n \in \mathbb{N}$ with I_f^n and I_f^m non-empty, if $n < m$ then $f_m < f_n$.

Now for $f, g \in \ell^p(I)$, suppose $f \prec g$ and $g \prec f$. Let $\phi_c : \mathbb{R} \to \mathbb{R}$ be a convex function defined by $\phi_c(x) = (x - c)\chi_{[c,\infty)}(x)$, with $c \in \mathbb{R}$. By Corollary 3.3,

$$\sum_{i \in I} \phi_c\big(f(i)\big) = \sum_{i \in I} \phi_c\big(g(i)\big) \tag{7}$$

for each $c \in \mathbb{R}$. For $c = 0$, we have

$$\sum_{i \in I_f^+} \phi_0\big(f(i)\big) = \sum_{i \in I} \phi_0\big(f(i)\big) = \sum_{i \in I} \phi_0\big(g(i)\big) = \sum_{i \in I_g^+} \phi_0\big(g(i)\big) \tag{8}$$

which shows that $I_f^+ \neq \emptyset$ if and only if $I_g^+ \neq \emptyset$. Suppose $I_f^+ \neq \emptyset$. Using induction, we show that for each $n \in \mathbb{N}$,

(i) $f_n = g_n$,

(ii) $|I_f^n| = |I_g^n|$.

For $n = 1$, if $I_f^1 = \emptyset$, then $f \leq 0$ and therefore $I_f^+ = \emptyset$ which is contrary to our assumption. Hence $I_f^1 \neq \emptyset$. Similarly $I_g^1 \neq \emptyset$. Suppose $f_1 \neq g_1$ and, for example, $f_1 < g_1$. Then for each $i_1 \in I_f^1$ and $i_2 \in I_g^1$, $f(i_1) = f_1 < g_1 = g(i_2)$. Using the convex function ϕ_c with $c = \min\{f_1, g_1\}$, we have

$$\sum_{i\in I} \phi_c\Big(f(i)\Big) = 0 < g_1 - f_1 \leq \sum_{i\in I} \phi_c\Big(g(i)\Big)$$

which contradicts (7). Hence $f_1 = g_1$. Again, taking $c = \max\{f_2, g_2\}$ in (7), we have

$$(f_1 - c)|I_f^1| = \sum_{i\in I} \phi_c\Big(f(i)\Big) = \sum_{i\in I} \phi_c\Big(g(i)\Big) = (g_1 - c)|I_g^1|.$$

Hence (i) and (ii) are satisfied for $n = 1$.
Suppose (i) and (ii) hold for each $k = 1, \ldots, n$. If $I_f^{n+1} = \emptyset$ then $I_f^j = \emptyset$ for all $j \geq n + 1$. Hence, using once more equation (8), we will have

$$\sum_{k=1}^{n} f_k |I_f^k| = \sum_{i\in I} \phi_0\Big(f(i)\Big) = \sum_{i\in I} \phi_0\Big(g(i)\Big) \geq \sum_{k=1}^{n+1} g_k |I_g^k|$$

which implies that the term $g_{n+1}|I_g^{n+1}|$ is non-positive. Hence $I_g^{n+1} = \emptyset$. In this case, $f_{n+1} = g_{n+1} = 0$, i.e. (i) and (ii) are satisfied for $n + 1$. If $I_f^{n+1} \neq \emptyset$ then the same argument shows that $I_g^{n+1} \neq \emptyset$. In this case, a similar procedure to that of $n = 1$, once with $c = \min\{f_{n+1}, g_{n+1}\}$ and then with $c = \max\{f_{n+2}, g_{n+2}\}$ in (7), implies (i) and (ii) for $n + 1$.

By (ii), there is a bijection $\theta_n : I_g^n \to I_f^n$ for each $n \in \mathbb{N}$ with $I_f^n \neq \emptyset$. Now we can define a bijection $\theta^+ : I_g^+ = \cup_{n\in\mathbb{N}} I_g^n \to I_f^+$ given by $\theta^+(j) = \theta_n(j)$ if $j \in I_g^n$.

Let D be a doubly stochastic operator on $\ell^p(I)$ satisfying $f = Dg$. For simplicity, we let $d_{ij} := \langle De_j, e_i\rangle$, for each $i, j \in I$. We show that if $I_f^n \neq \emptyset$ then

$$\forall i \in I_f^n, \qquad \sum_{j\in I_g^n} d_{ij} = 1 \tag{9}$$

and

$$\forall j \in I_g^n, \qquad \sum_{i\in I_f^n} d_{ij} = 1 \tag{10}$$

To prove (9) and (10), first suppose $n = 1$ and that $I_f^1 \neq \emptyset$. We show that $\sum_{j\notin I_g^1} d_{ij} = 0$, for all $i \in I_f^1$, which then implies (9) for $n = 1$. If for some $i \in I_f^1$, $\sum_{j\notin I_g^1} d_{ij} > 0$, then

$$0 < f_1 = f(i) = \sum_{j\in I} d_{ij} g(j) = \sum_{j\in I_g^1} d_{ij} g_1 + \sum_{j\notin I_g^1} d_{ij} g(j) < \sum_{j\in I_g^1} d_{ij} g_1 + \sum_{j\notin I_g^1} d_{ij} g_1 = g_1$$

which contradicts the fact that $f_1 = g_1$. Hence we have shown that $d_{ij} = 0$, for each $i \in I_f^1$ and $j \notin I_g^1$, and therefore, $\sum_{j\in I_g^1} d_{ij} = 1$.

To see (10) for $n=1$, suppose there exists $j \in I_g^1$ with $\sum_{i\in I_f^1} d_{ij} < 1$, then

$$|I_f^1| = \sum_{i\in I_f^1}\sum_{j\in I_g^1} d_{ij} = \sum_{j\in I_g^1}\sum_{i\in I_f^1} d_{ij} < |I_g^1|,$$

which contradicts (ii) for $n=1$.
By induction and using a similar method, we will see that (9) and (10) hold for each $n \in \mathbb{N}$.

An immediate consequence of the above facts is that,

(i) $\forall i \in I_f^+ \;\; \forall j \in I \setminus I_g^+, \quad d_{ij} = 0$

(ii) $\forall j \in I_g^+ \;\; \forall i \in I \setminus I_f^+, \quad d_{ij} = 0$

Replacing f and g by $-f$ and $-g$ and noting that $I_f^- = I_{-f}^+$ and $I_g^- = I_{-g}^+$, we obtain a similar bijection $\theta^- : I_g^- \to I_f^-$, and the following results.

(iii) $\forall i \in I_f^- \;\; \forall j \in I \setminus I_g^-, \quad d_{ij} = 0$

(iv) $\forall j \in I_g^- \;\; \forall i \in I \setminus I_f^-, \quad d_{ij} = 0$

Using all above facts, it is easily verified that if $D_0 : \ell^p(I_g^0) \to \ell^p(I_f^0)$ is the map defined by $\langle D_0 e_j, e_i\rangle = d_{ij}$, for $i \in I_f^0$ and $j \in I_g^0$, then it is doubly stochastic. By Theorem 2.2, $|I_f^0| = |I_g^0|$, i.e. there exists a bijection $\theta^0 : I_g^0 \to I_f^0$. Now we define a bijection $\theta : I \to I$ by

$$\theta(j) = \begin{cases} \theta^+(j) & j \in I_g^+ \\ \theta^-(j) & j \in I_g^- \\ \theta^0(j) & j \in I_g^0 \end{cases}$$

Let P be the permutation on $\ell^p(I)$ corresponding to θ . We claim that $f = Pg$. To see this, note that for each $i \in I$,

$$\big(Pg\big)(i) = \langle \sum_{j\in I} g(j) e_{\theta(j)}, e_i\rangle(i) = g\big(\theta^{-1}(i)\big)$$

If $i \in I_f^+$ then there exists $n \in \mathbb{N}$ such that $i \in I_f^n$, and therefore $\theta^{-1}(i) \in I_g^n$. Hence $g\big(\theta^{-1}(i)\big) = g_n = f_n = f(i)$. A similar argument holds if $i \in I_f^-$. Finally, if $i \in I_f^0$, then $\theta^{-1}(i) \in I_g^0$. Hence $g\big(\theta^{-1}(i)\big) = 0 = f(i)$. Thus, $Pg(i) = f(i)$, for all $i \in I$, i.e. $f = Pg$.

The converse is evident. □

4 Linear Maps Preserving Majorization

In this section, we characterize bounded linear operators on $\ell^p(I)$, with $p \in (1, +\infty)$ and I an infinite set, which preserves the majorization relation.

Definition 4.1 *A bounded linear operator $T : \ell^p(I) \to \ell^p(I)$ is called a majorization preserver on $\ell^p(I)$, if T preserves the majorization relation, i.e. for $f, g \in \ell^p(I)$, $f \prec g$ implies $Tf \prec Tg$. We denote by $\mathcal{M}_{_{Pr}}\big(\ell^p(I)\big)$ the set of all such operators.*

In order to have some examples of this class of operators, we need first some preliminaries. It is easily seen that for $\alpha \in \mathbb{R}$ and $S, T \in \mathcal{M}_{_{Pr}}\big(\ell^p(I)\big)$, $\alpha T, ST \in \mathcal{M}_{_{Pr}}\big(\ell^p(I)\big)$, i.e. $\mathcal{M}_{_{Pr}}\big(\ell^p(I)\big)$ is closed under the scalar multiplication and combination. We will see later that this set is not closed under addition.

For a one-to-one map $\sigma : I \to I$, let $P_\sigma : \ell^p(I) \to \ell^p(I)$ be defined for each for $f = \sum_{j\in I} f_j e_j \in \ell^p(I)$ by $P_\sigma(f) = \sum_{j\in I} f_j e_{\sigma(j)}$. Clearly, P_σ is a bounded linear operator with $\|P_\sigma\| \leq 1$. Note that if, in addition, $\sigma : I \to I$ is on-to then P_σ is a permutation.

Lemma 4.2 *Let $D : \ell^p(I) \to \ell^p(I)$ be a doubly stochastic operator and Σ be any family of one-to-one maps from I to I which satisfies $\sigma_1(I) \cap \sigma_2(I) = \emptyset$, for distinct $\sigma_1, \sigma_2 \in \Sigma$. Then there exists a doubly stochastic $\widetilde{D} \in \mathcal{DS}\big(\ell^p(I)\big)$ such that $P_\sigma D = \widetilde{D} P_\sigma$, for all $\sigma \in \Sigma$.*

Proof. For $i, j \in I$, let $d_{ij} := \langle De_j, e_i \rangle$ and suppose $\tilde{d}_{ij}$ is defined by

$$\tilde{d}_{ij} = \begin{cases} d_{\sigma^{-1}(i)\sigma^{-1}(j)} & \text{if} \quad i, j \in \sigma(I) \quad (\text{ for some } \sigma \in \Sigma) \\ 0 & \text{if} \quad i \in \sigma(I),\ j \notin \sigma(I)\ (\text{ for some } \sigma \in \Sigma) \\ 1 & \text{if} \quad i \notin \cup_{\sigma\in\Sigma}\sigma(I),\ j = i \\ 0 & \text{if} \quad i \notin \cup_{\sigma\in\Sigma}\sigma(I),\ j \neq i \end{cases} \tag{11}$$

By considering the two cases $i \in \sigma(I)$, for some $\sigma \in \Sigma$, and $i \notin \cup_{\sigma\in\Sigma}\sigma(I)$, it is easy to see that $\sum_{j\in I} \tilde{d}_{ij} = 1$ for each $i \in I$. Similarly, writing (11) in the following form,

$$\tilde{d}_{ij} = \begin{cases} d_{\sigma^{-1}(i)\sigma^{-1}(j)} & \text{if} \quad j, i \in \sigma(I)\ (\text{ for some } \sigma \in \Sigma) \\ 0 & \text{if} \quad j \in \sigma(I),\ i \notin \sigma(I)\ (\text{ for some } \sigma \in \Sigma) \\ 1 & \text{if} \quad j \notin \cup_{\sigma\in\Sigma}\sigma(I),\ i = j \\ 0 & \text{if} \quad j \notin \cup_{\sigma\in\Sigma}\sigma(I),\ i \neq j, \end{cases}$$

it is seen that $\sum_{i\in I} \tilde{d}_{ij} = 1$ for each $j \in I$. Hence, using Proposition 2.6, there exists a doubly stochastic operator $\widetilde{D} : \ell^p(I) \to \ell^p(I)$ which satisfies $\langle \widetilde{D} e_j, e_i \rangle = \tilde{d}_{ij}$ for all $i, j \in I$.

It remains to show that $P_\sigma D = \widetilde{D} P_\sigma$, for each $\sigma \in \Sigma$. We have

$$\widetilde{D} P_\sigma(e_j) = \widetilde{D}(e_{_{\sigma(j)}}) = \sum_{i\in\sigma(I)} \tilde{d}_{i\sigma(j)} e_i = \sum_{i\in\sigma(I)} d_{\sigma^{-1}(i)j} e_i = \sum_{r\in I} d_{rj} e_{_{\sigma(r)}}$$

and

$$P_\sigma D(e_j) = P_\sigma\Big(\sum_{i\in I} d_{ij} e_i\Big) = \sum_{i\in I} d_{ij} e_{_{\sigma(i)}}.$$

Hence

$$P_\sigma D(e_j) = \widetilde{D} P_\sigma(e_j),$$

for all $j \in I$. Thus $\widetilde{D} P_\sigma = P_\sigma D$, for each $\sigma \in \Sigma$. □

Example 4.3 Let $\sigma : I \to I$ be a one-to-one map. For $f, g \in \ell^p(I)$ suppose $f \prec g$, i.e. $f = Dg$ for some $D \in \mathcal{DS}\big(\ell^p(I)\big)$. By Lemma 4.2, corresponding to the singleton $\Sigma = \{\sigma\}$, there exists $\widetilde{D} \in \mathcal{DS}\big(\ell^p(I)\big)$ for which $P_\sigma D = \widetilde{D} P_\sigma$. Therefore $P_\sigma f = P_\sigma D g = \widetilde{D} P_\sigma g$, which shows that $P_\sigma f \prec P_\sigma g$. Thus each P_σ preserves majorization. In particular each permutation belongs to $\mathcal{M}_{_{Pr}}\big(\ell^p(I)\big)$, i.e.

$$\mathcal{P}\big(\ell^p(I)\big) \subseteq \mathcal{M}_{_{Pr}}\big(\ell^p(I)\big).$$

Example 4.4 For a fixed $k \in \mathbb{N}$, let $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ be an operator defined for $f = \sum_{n=1}^\infty f_n e_n \in \ell^p(\mathbb{N})$ by $T(f) = \sum_{n=1}^\infty f_{[\frac{n}{k}]} e_n$, where $[\frac{n}{k}]$ denotes the greatest integer contained in $\frac{n}{k}$, and $f_0 := 0$. Then, T is easily seen to be linear and bounded (with $\|T\| = \sqrt[p]{k}$). Suppose $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ where each $\sigma_i : \mathbb{N} \to \mathbb{N}$ (for $1 \leq i \leq k$) is a one-to-one map defined by $\sigma_i(n) = nk + i - 1$, for all $n \in \mathbb{N}$. It is easy to see that $T = \sum_{i=1}^k P_{\sigma_i}$ and that the family Σ satisfies condition of Lemma 4.2. If $f \prec g$ in $\ell^p(I)$, i.e. $f = Dg$ for some $D \in \mathcal{DS}\big(\ell^p(\mathbb{N})\big)$, then Lemma 4.2 implies that there exists a doubly stochastic $\tilde{D} \in \mathcal{DS}\big(\ell^p(\mathbb{N})\big)$ for which $\tilde{D} P_{\sigma_i} = P_{\sigma_i} D$ for $i = 1, \ldots, k$. Hence

$$Tf = \sum_{i=1}^k P_{\sigma_i} f = \sum_{i=1}^k P_{\sigma_i} D g = \sum_{i=1}^k \tilde{D} P_{\sigma_i} g = \tilde{D} T g$$

i.e. T preserves majorization.

In the following theorem, which is a generalization of Example 4.4, we construct a family of bounded linear operators which preserve majorization. As we will see in Theorem 4.9, in the case $1 < p < +\infty$, every majorization preserver will also be in this form.

Theorem 4.5 *Let $p \in [1, +\infty)$, I be an infinite set and $I_0 \subset I$ be a countable subset. Moreover, suppose $\Sigma = \{\sigma_i : I \to I\ ;\ i \in I_0\}$ is a family of one-to-one maps such that for all $i_1, i_2 \in I_0$ with $i_1 \neq i_2$, $\sigma_{i_1}(I) \cap \sigma_{i_2}(I) = \emptyset$. If $(\alpha_i)_{i \in I_0}$ is an element of $\ell^p(I_0)$ then $T := \sum_{i \in I_0} \alpha_i P_{\sigma_i}$ is a majorization preserver.*

Proof. It is easily seen that $T = \sum_{i \in I_0} \alpha_i P_{\sigma_i}$ is a well-defined bounded linear map. Suppose $f \prec g$, for $f, g \in \ell^p(I)$, and therefore $f = Dg$ for some $D \in \mathcal{DS}\big(\ell^p(I)\big)$.
Corresponding to the family $\{\sigma_i : I \to I;\ i \in I_0\}$, let $\widetilde{D} \in \mathcal{DS}\big(\ell^p(I)\big)$ be the operator given by Lemma 4.2. Then

$$\begin{aligned}
\widetilde{D}(Tg) &= \widetilde{D}\Big(\sum_{i \in I_0} \alpha_i P_{\sigma_i}(g)\Big) \\
&= \sum_{i \in I_0} \alpha_i \widetilde{D} P_{\sigma_i}(g) \\
&= \sum_{i \in I_0} \alpha_i P_{\sigma_i} D(g) \\
&= \sum_{i \in I_0} \alpha_i P_{\sigma_i}(f) \\
&= Tf
\end{aligned}$$

Hence $Tf \prec Tg$. $\square$

As was pointed out, the converse of this theorem is also true for $p \in (1, +\infty)$. In order to prove it, we need some lemmas.

Lemma 4.6 *Let $a, b \in \mathbb{R}$ and $\{a_i; i \in I\}$, $\{b_i; i \in I\}$ be two families of real numbers, where I is assumed to be a countable indexed set. If*

$$\alpha a + \beta b \in \{\alpha a_i + \beta b_i; i \in I\},$$

for all $\alpha, \beta \in \mathbb{R}$, then there exists $i \in I$ such that $a = a_i$ and $b = b_i$.

Proof. Let $C := \{(\alpha, \beta) \in \mathbb{R}^2 \ ; \ \alpha, \beta > 0, \ \alpha^2 + \beta^2 = 1\}$. Then, by assumption, for each $(\alpha, \beta) \in C$ there exists $i = i_{(\alpha,\beta)} \in I$ for which

$$\alpha a + \beta b = \alpha a_i + \beta b_i$$

Since I is countable and C is uncountable there exists two distinct elements $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in C$ with $i_{(\alpha_1,\beta_1)} = i_{(\alpha_2,\beta_2)}$, which for simplicity we denote it by i itself. Hence

$$\begin{cases} \alpha_1 a + \beta_1 b = \alpha_1 a_i + \beta_1 b_i \\ \\ \alpha_2 a + \beta_2 b = \alpha_2 a_i + \beta_2 b_i \end{cases} \tag{12}$$

Note that any two distinct elements of C are linearly independent. Hence (12) implies that $a = a_i$ and $b = b_i$. $\square$

As the following lemma shows, if $T : \ell^p(I) \to \ell^p(I)$ is a linear majorization preserver then, roughly speaking, each row of T contains, at most, one non-zero element. In what follows, for $f, g \in \ell^p(I)$, we use the notation $f \sim g$ whenever each of f and g is majorized by the other i.e. $f \prec g$ and $g \prec f$.

Lemma 4.7 *Let I be any infinite set, $p \in (1, \infty)$, and $T \in \mathcal{M}_{Pr}\big(\ell^p(I)\big)$. Then for any $i \in I$, there is at most one $j \in I$ such that $\langle Te_j, e_i\rangle \neq 0$.*

Proof. Suppose, on the contrary, there exists $i_1 \in I$ and two different elements $j_1, j_2 \in I$ for which

$$\langle Te_{j_1}, e_{i_1}\rangle \neq 0 \ , \ \langle Te_{j_2}, e_{i_1}\rangle \neq 0$$

For simplicity we denote $\langle Te_{j_1}, e_{i_1}\rangle, \langle Te_{j_2}, e_{i_1}\rangle$, respectively, by a, b. Let F be given by

$$F = \{i \in I; \langle Te_{j_1}, e_i\rangle = a\}.$$

Then F is non-empty, and the inequality

$$\begin{aligned} \sum_{i \in F} |a|^p \quad &= \sum_{i \in F} |\langle Te_{j_1}, e_i\rangle|^p \\ &\leq \sum_{i \in I} |\langle Te_{j_1}, e_i\rangle|^p \\ &= \|Te_{j_1}\|^p < \infty \end{aligned}$$

shows that F is finite. On the other hand, for any given $j \neq j_1$ and $\alpha, \beta \in \mathbb{R}$, since $\alpha e_{j_1} + \beta e_{j_2} \sim \alpha e_{j_1} + \beta e_j$, we have

$$\forall \alpha, \beta \in \mathbb{R} \quad \alpha T e_{j_1} + \beta T e_{j_2} \sim \alpha T e_{j_1} + \beta T e_j$$

Hence, by Theorem 3.5,

$$\alpha a + \beta b \in \{\alpha \langle T e_{j_1}, e_i \rangle + \beta \langle T e_j, e_i \rangle \ ; \ i \in I\}$$

for $j \neq j_1$ and all $\alpha, \beta \in \mathbb{R}$. But the indexed set I can be replaced by a countable one. Hence, by Lemma 4.6, for each $j \in I \setminus \{j_1\}$ there exists $i \in I$ such that

$$\langle T e_{j_1}, e_i \rangle = a, \ \langle T e_j, e_i \rangle = b.$$

Thus $i \in F$. Since I is infinite and F is finite, there exists $i_0 \in F$ and a sequence (j_n) in I, with $j_m \neq j_n$ for $m \neq n$, such that

$$\langle e_{j_n}, T^* e_{i_0} \rangle = \langle T e_{j_n}, e_{i_0} \rangle = b \neq 0$$

for all $n \in \mathbb{N}$. This contradicts the fact that e_{j_n} converges to 0 in the weak topology of $\ell^p(I)$.
□

Using the previous lemma, the next example shows that the sum of two majorization preservers need not be a preserver.

Example 4.8 Let $\sigma_1, \sigma_2 : \mathbb{N} \to \mathbb{N}$ be defined by $\sigma_1(n) = 2n, \ \sigma_2(n) = n$, for each $n \in \mathbb{N}$. Then by Example 4.3, the maps P_{σ_1} and P_{σ_2} are both majorization preservers. Now suppose $T := P_{\sigma_1} + P_{\sigma_2}$. Then, since

$$\langle T e_1, e_2 \rangle = \langle T e_2, e_2 \rangle = 1,$$

by Lemma 4.7, T is no longer a majorization preserver.

We now have the main result of this paper.

Theorem 4.9 *Suppose I is an infinite set and $p \in (1, +\infty)$. For a bounded linear operator T on $\ell^p(I)$ the following conditions are equivalent.*

(i) *T is a majorization preserver.*

(ii) *For $f, g \in \ell^p(I)$, if $f \sim g$ then $Tf \sim Tg$. Furthermore, for any $i \in I$ there is at most one $j \in I$ for which $\langle T e_j, e_i \rangle \neq 0$.*

(iii) *For any $j_1, j_2 \in I$, $T e_{j_1} \sim T e_{j_2}$ and for each $i \in I$ there is at most one $j \in I$ with $\langle T e_j, e_i \rangle \neq 0$.*

(iv) *$T = \sum_{i \in I_0} \alpha_i P_{\sigma_i}$, where I_0 a countable subset of I, $(\alpha_i)_{i \in I_0}$ is an element of $\ell^p(I_0)$, and $\{\sigma_i : I \to I \ ; \ i \in I_0\}$ is a family of one-to-one maps such that for all $i_1, i_2 \in I_0$ with $i_1 \neq i_2$, $\sigma_{i_1}(I) \cap \sigma_{i_2}(I) = \emptyset$.*

Proof. Suppose T is a non-zero bounded linear operator on $\ell^p(I)$.
(i)$\Rightarrow$ (ii) is obtained from Definition 4.1 and Lemma 4.7.
(ii)$\Rightarrow$ (iii) is clear.
(iii)$\Rightarrow$ (iv). For $j \in I$ let $I(j) := \{i \in I; \langle Te_j, e_i\rangle \neq 0\}$. According to (iii) for $j_1 \neq j_2$,

$$I(j_1) \bigcap I(j_2) = \emptyset. \tag{13}$$

On the other hand, $T \neq 0$. So there exists $j_0 \in I$ such that $Te_{j_0} \neq 0$. Hence $I(j_0) \neq \emptyset$. Now for $j \in I$ with $j \neq j_0$, $Te_j \sim Te_{j_0}$. Let $P_j : \ell^p(I) \to \ell^p(I)$ be the permutation given by Theorem 3.5, so that $Te_j = P_j Te_{j_0}$. Also let $\theta_j : I \to I$ be the bijection corresponds to P_j which is uniquely determined by $P_j(e_i) = e_{\theta_j(i)}$, for all $i \in I$.
Let $I_0 := I(j_0)$ which is obviously a countable subset of I, $\sigma_i : I \to I$ be defined by $\sigma_i(j) = \theta_j(i)$, and $\alpha_i := \langle Te_{j_0}, e_i\rangle$, for $i \in I_0$.
Note that for $i, j \in I$

$$\begin{aligned} \langle Te_j, e_{\theta_j(i)}\rangle &= \langle Te_j, P_j(e_i)\rangle \\ &= \langle P_j^* Te_j, e_i\rangle \\ &= \langle P_j^{-1} Te_j, e_i\rangle. \end{aligned}$$

Since $P^{-1}Te_j = Te_{j_0}$, we have

$$\langle Te_j, e_{\theta_j(i)}\rangle = \langle Te_{j_0}, e_i\rangle. \tag{14}$$

for every $i, j \in I$. This shows that for each $i \in I_0 = I(j_0)$, $\theta_j(i) \in I(j)$. Hence for $i \in I_0$ and $j_1 \neq j_2$, since $\sigma_i(j_1) = \theta_{j_1}(i) \in I(j_1)$, and $\sigma_i(j_2) = \theta_{j_2}(i) \in I(j_2)$, (13) shows that $\sigma_i(j_1) \neq \sigma_i(j_2)$, i.e. $\sigma_i : I \to I$ is one-to-one.

Let i_1, i_2 are two distinct elements of I_0. We will show that σ_{i_1} and σ_{i_2} have disjoint ranges. Suppose, on the contrary, there exist $j_1, j_2 \in I$ for which $\sigma_{i_1}(j_1) = \sigma_{i_2}(j_2)$ which implies that

$$\theta_{j_1}(i_1) = \theta_{j_2}(i_2). \tag{15}$$

By (14), we have

$$\langle Te_{j_1}, e_{\theta_{j_1}(i_1)}\rangle = \langle Te_{j_0}, e_{i_1}\rangle \neq 0, \tag{16}$$

and

$$\langle Te_{j_2}, e_{\theta_{j_2}(i_2)}\rangle = \langle Te_{j_0}, e_{i_2}\rangle \neq 0. \tag{17}$$

By (15), $e_{\theta_{j_1}(i_1)} = e_{\theta_{j_2}(i_2)}$. Hence (16), (17) and the assumption of (iii) implies that $j_1 = j_2$, which, again by (15), leads to the contradiction $i_1 = i_2$.

Finally, we show that $\sum_{i \in I_0} \alpha_i P_{\sigma_i}$ converges (unconditionally) in norm to T. First we consider the case where I_0 is infinite. For simplicity, suppose $I_0 = \mathbb{N}$. We will show that

$\sum_{n=1}^{\infty} \alpha_n P_{\sigma_n}$ converges to T in the norm topology of $\mathcal{B}\Big(\ell^p(I)\Big)$. For $j \in I$, we have

$$\begin{aligned} Te_j = P_j(Te_{j_0}) &= P_j(\sum_{i\in I_0} \langle Te_{j_0}, e_i\rangle e_i) \\ &= \sum_{i\in I_0} \langle Te_{j_0}, e_i\rangle P_j e_i \\ &= \sum_{i\in I_0} \alpha_i e_{\sigma_i(j)} \\ &= \sum_{n=1}^{\infty} \alpha_n e_{\sigma_n(j)} \end{aligned}$$

Hence for $f = \sum_{j\in I} f_j e_j \in \ell^p(I)$, and $n \in \mathbb{N}$,

$$\begin{aligned} \|Tf - \sum_{k=1}^{n} \alpha_k P_{\sigma_k}(f)\|^p &= \|\sum_{j\in I} f_j Te_j - \sum_{k=1}^{n}\sum_{j\in I} \alpha_k f_j \ e_{\sigma_k(j)}\|^p \\ &= \|\sum_{j\in I}\sum_{k\in\mathbb{N}} \alpha_k f_j \ e_{\sigma_k(j)} - \sum_{k=1}^{n}\sum_{j\in I} \alpha_k f_j \ e_{\sigma_k(j)}\|^p \\ &= \|\sum_{k>n, j\in I} \alpha_k f_j \ e_{\sigma_k(j)}\|^p \\ &= \sum_{k>n, j\in I} |\alpha_k f_j|^p = \|f\|^p \sum_{k=n+1}^{\infty} |\alpha_k|^p \end{aligned}$$

Hence $\|T - \sum_{k=1}^{n} \alpha_k P_{\sigma_k}\| \leq \big(\sum_{k=n+1}^{\infty} |\alpha_k|^p\big)^{\frac{1}{p}} \to 0$, as $n \to \infty$.
(iv)$\Rightarrow$(i). This is Theorem 4.5. □

By Theorem 4.5, if $T : \ell^1(I) \to \ell^1(I)$ is in the form described in part (iv) of the previous theorem, then T is a majorization preserver. However it should be noted that, as the following example shows, not every majorization preserver $T : \ell^1(I) \to \ell^1(I)$ is necessarily in this form.

Example 4.10 Let $h = \sum_{j\in I} h_j e_j \in \ell^1(I)$ be a non-zero element and suppose $T_h : \ell^1(I) \to \ell^1(I)$ is defined, for each $f = \sum_{i\in I} f_i \in \ell^1(I)$, by $T_h(f) = \big(\sum_{i\in I} f_i\big)h$. It is easily seen that T_h is linear and bounded (with $\|T_h\| = \|h\|$). On the other hand, for $f, g \in \ell^1(I)$, if $f \prec g$ then $\sum_{i\in I} f_i = \sum_{i\in I} g_i$. Hence $T_h(f) = T_h(g)$, which clearly implies that $T_h(f) \prec T_h(g)$. Thus T_h is a majorization preserver, which is not in the form described in the previous theorem.

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